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Some inequalities via probabilistic method

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ABSTRACT

In this paper, we obtain some new inequalities by means of the mean inequalities of random variables, which include generalizations of the *Greub–Rheinboldt* inequality.

1. Introduction and main results

Probabilistic method is a useful tool in the study of inequalities, which are fundamental to many fields including mathematics, statistics, physics and economics. There are a large number of works available in the literature. For example, Shaked, Tong, Shanthikumar and Wang give some useful results [1–8]. In this paper, we use probabilistic method to derive some new inequalities, which include the extensions of the *Greub–Rheinboldt* inequality.

We begin by introducing some preliminary concepts and known results which can also be found in [6,7].

Definition 1.1. The *supremum and infimum of the random variable* ξ are defined as $\inf_x\{x : P(\xi \le x) = 1\}$ and $\sup_x\{x : P(\xi \ge x) = 1\}$, respectively, and denoted by $\sup\xi$ and $\inf\xi$.

Definition 1.2. If ξ is bounded, the *arithmetic mean of the random variable* ξ, $A(\xi)$, is given by

$$A(\xi) = \frac{\sup\xi + \inf\xi}{2}.$$

In addition, if $\inf\xi \ge 0$, we define the *geometric mean of the random variable* ξ, $G(\xi)$, to be

$$G(\xi) = \sqrt{\sup\xi \cdot \inf\xi}.$$

Definition 1.3. If $\xi_1, \ldots, \xi_n$ are bounded random variables, the *independent arithmetic mean of the product of random variables* $\xi_1, \ldots, \xi_n$, $\overline{A}(\xi_1, \ldots, \xi_n)$, is given by

$$\overline{A}(\xi_1, \ldots, \xi_n) = \frac{\prod_{i=1}^{n}\sup\xi_i + \prod_{i=1}^{n}\inf\xi}{2}.$$

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Definition 1.4. If $\xi_1, \dots, \xi_n$ are bounded random variables with $\inf \xi_i \geq 0, i = 1, \dots, n$, we define the *independent geometric mean of the product of random variables* $\xi_1, \dots, \xi_n$ to be

$$\overline{G}(\xi_1, \dots, \xi_n) = \sqrt{\prod_{i=1}^{n} \sup \xi_i \inf \xi_i}.$$

Remark 1.5. If $\xi_1, \dots, \xi_n$ are independent, then

$$\overline{A}(\xi_1, \dots, \xi_n) = A\left(\prod_{i=1}^{n} \xi_i\right),$$

$$\overline{G}(\xi_1, \dots, \xi_n) = G\left(\prod_{i=1}^{n} \xi_i\right).$$

In [7], the author proves the following mean inequality of random variables: Let $n \geq 2$ be a positive integer and $\xi_1, \dots, \xi_n$ be bounded random variables. If $\inf \xi_i > 0, i = 1, \dots, n$, then

$$\frac{\prod_{k=1}^{n} E\xi_k^2}{E^2\left(\prod_{k=1}^{n} \xi_k\right)} \leq \prod_{k=2}^{n} \frac{\overline{A}^2(\xi_1, \dots, \xi_k)}{\overline{G}^2(\xi_1, \dots, \xi_k)}. \tag{1.1}$$

The main purpose of the present paper is to establish the following inequality of random variables:

Theorem 1.6. *Let* $\xi_1, \dots, \xi_n$ *and* $\eta_1, \dots, \eta_m$ *be bounded random variables and* $\inf \xi_i > 0, \inf \eta_j > 0$, *for* $i = 1, \dots, n$ *and* $j = 1, \dots, m$. *Suppose that*

$$\frac{\prod_{i=1}^{n} E\xi_i^2}{E^2\left(\prod_{i=1}^{n} \xi_i\right)} \leq U(n), \qquad \frac{\prod_{j=1}^{m} E\eta_j^2}{E^2\left(\prod_{j=1}^{m} \eta_j\right)} \leq Q(m), \tag{1.2}$$

then

$$\frac{\prod_{i=1}^{n} E\xi_i^2 \prod_{j=1}^{m} E\eta_j^2}{E^2\left(\prod_{i=1}^{n} \xi_i \prod_{j=1}^{m} \eta_j\right)} \leq \frac{\overline{A}^2(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m)}{\overline{G}^2(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m)} U(n)Q(m). \tag{1.3}$$

This result can, in turn, be extended to establish other new inequalities, which include generalizations of the *Greub–Rheinboldt* inequality [9].

2. The proof of the inequality

In order to prove the inequality (1.3), we need the following known lemma which we state here without proof.

Lemma 2.1. *If* $0 < m_2 \leq m_1 \leq M_1 \leq M_2$, *then*

$$\frac{\frac{1}{2}(m_1 + M_1)}{\sqrt{m_1 M_1}} \leq \frac{\frac{1}{2}(m_2 + M_2)}{\sqrt{m_2 M_2}}.$$

Now we give the proof of the inequality (1.3).

Proof. Let

$$A_i = \sup \xi_i, \qquad a_i = \inf \xi_i, \qquad B_j = \sup \eta_j, \qquad b_j = \inf \eta_j, \quad i = 1, \dots, n;\ j = 1, \dots, m.$$

It is easy to see

$$P\left\{\left(\prod_{i=1}^{n} \xi_i \prod_{j=1}^{m} B_j - \prod_{i=1}^{n} a_i \prod_{j=1}^{m} \eta_j\right)\left(\prod_{i=1}^{n} A_i \prod_{j=1}^{m} \eta_j - \prod_{i=1}^{n} \xi_i \prod_{j=1}^{m} b_j\right) \geq 0\right\} = 1. \tag{2.1}$$

So

$$P\left\{\left(\prod_{i=1}^{n}A_i\prod_{j=1}^{m}B_j+\prod_{i=1}^{n}a_i\prod_{j=1}^{m}b_j\right)\prod_{i=1}^{n}\xi_i\prod_{j=1}^{m}\eta_j\geq\prod_{i=1}^{n}A_ia_i\prod_{j=1}^{m}\eta_j^2+\prod_{j=1}^{m}B_jb_j\prod_{i=1}^{n}\eta_i^2\right\}=1. \tag{2.2}$$

Using (1.2), we have

$$\begin{aligned}
&(A_1\cdots A_nB_1\cdots B_m+a_1\cdots a_nb_1\cdots b_m)E(\xi_1\cdots\xi_n\eta_1\cdots\eta_m)\\
&\quad\geq A_1\cdots A_na_1\cdots a_nE(\eta_1^2\cdots\eta_m^2)+B_1\cdots B_mb_1\cdots b_mE(\xi_1^2\cdots\xi_n^2)\\
&\quad\geq A_1\cdots A_na_1\cdots a_nE^2(\eta_1\cdots\eta_m)+B_1\cdots B_mb_1\cdots b_mE^2(\xi_1\cdots\xi_n)\\
&\quad\geq A_1\cdots A_na_1\cdots a_n\frac{E\eta_1^2\cdots E\eta_m^2}{Q(m)}+B_1\cdots B_mb_1\cdots b_m\frac{E\xi_1^2\cdots E\xi_n^2}{U(n)}\\
&\quad\geq 2\left\{A_1\cdots A_na_1\cdots a_n\frac{E\eta_1^2\cdots E\eta_m^2}{Q(m)}B_1\cdots B_mb_1\cdots b_m\frac{E\xi_1^2\cdots E\xi_n^2}{U(n)}\right\}^{1/2}.
\end{aligned} \tag{2.3}$$

Therefore

$$\left\{\frac{\overline{G}^2(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m)\prod\limits_{i=1}^{n}E\xi_i^2\prod\limits_{j=1}^{m}E\eta_j^2}{U(n)Q(m)}\right\}^{1/2}\leq\overline{A}(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m)E\left(\prod_{i=1}^{n}\xi_i\prod_{j=1}^{m}\eta_j\right), \tag{2.4}$$

from which the result of our theorem follows. □

In combination with (1.1), we can state the following additional result:

Corollary 1. *Let $\xi_1,\ldots,\xi_n$ and $\eta_1,\ldots,\eta_m$ be bounded random variables, with $\inf\xi_i>0$, $\inf\eta_j>0$, $i=1,\ldots,n$ and $j=1,\ldots,m$ for $n,m\geq 2$, then*

$$\frac{\prod\limits_{i=1}^{n}E\xi_i^2\prod\limits_{j=1}^{m}E\eta_j^2}{E^2\left(\prod\limits_{i=1}^{n}\xi_i\prod\limits_{j=1}^{m}\eta_m\right)}\leq\frac{\overline{A}^2(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m)}{\overline{G}^2(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m)}\prod_{k=1}^{n}\frac{\overline{A}^2(\xi_1,\ldots,\xi_k)}{\overline{G}^2(\xi_1,\ldots,\xi_k)}\prod_{k=1}^{m}\frac{\overline{A}^2(\eta_1,\ldots,\eta_k)}{\overline{G}^2(\eta_1,\ldots,\eta_k)}. \tag{2.5}$$

3. Some new inequalities

In this section, we exhibit some of the applications of the inequalities (1.3). First, we give the following inequality.

Theorem 3.1. *Let*

$$\begin{aligned}
&a_{ij}>0, &&a_i=\min_j a_{ij}, &&A_i=\max_j a_{ij}, &&i=1,\ldots,n;\ j=1,\ldots,m,\\
&b_{ij}>0, &&b_i=\min_j b_{ij}, &&B_i=\max_j b_{ij}, &&i=1,\ldots,t;\ j=1,\ldots,m,
\end{aligned}$$

then

$$\begin{aligned}
\prod_{i=1}^{n}\sum_{j=1}^{m}a_{ij}^2\cdot\prod_{i=1}^{t}\sum_{j=1}^{m}b_{ij}^2\leq\frac{m^{n+t-2}}{4^{n+t-1}}\cdot\frac{\left[\prod\limits_{i=1}^{n}a_i\prod\limits_{i=1}^{t}b_i+\prod\limits_{i=1}^{n}A_i\prod\limits_{i=1}^{t}B_i\right]^2}{\prod\limits_{i=1}^{n}a_iA_i\prod\limits_{i=1}^{t}b_iB_i}\prod_{k=2}^{n}\frac{(a_1\cdots a_k+A_1\cdots A_k)^2}{a_1\cdots a_kA_1\cdots A_k}\\
\times\prod_{k=2}^{t}\frac{(b_1\cdots b_k+B_1\cdots B_k)^2}{b_1\cdots b_kB_1\cdots B_k}\left(\sum_{j=1}^{m}\prod_{i=1}^{n}a_{ij}b_{ij}\right)^2.
\end{aligned} \tag{3.1}$$

Proof. We define the random vector (ξ_1, η_1) by the joint probability density function

$$P(\xi_1 = a_{1i}, \eta_1 = b_{1j}) = \begin{cases} \frac{1}{m}, & i = j, \\ 0, & \text{otherwise} \end{cases} \qquad i, j = 1, \dots, m.$$

We also define

$$f_i(a_{1j}) = a_{ij}, \qquad \xi_i = f_i(\xi_1), \quad i = 2, \dots, n; \ j = 1, \dots, m.$$

Then

$$E\xi_i^2 = \frac{1}{m}\sum_{j=1}^{m} a_{ij}^2, \quad i = 1, \dots, n,$$

$$E(\xi_1 \cdots \xi_n) = \frac{1}{m}\sum_{j=1}^{m}\prod_{i=1}^{n} a_{ij},$$

$$\overline{A}(\xi_1, \dots, \xi_k) = \frac{1}{2}(a_1 \cdots a_k + A_1 \cdots A_k),$$

$$\overline{G}(\xi_1, \dots, \xi_k) = \sqrt{a_1 \cdots a_k A_1 \cdots A_k}.$$

Using inequality (1.1) gives

$$\frac{E\xi_1^2 \cdots E\xi_n^2}{E^2(\xi_1 \cdots \xi_n)} \le \prod_{k=2}^{n} \frac{\left[\frac{1}{2}(a_1 \cdots a_k + A_1 \cdots A_k)\right]^2}{\left[\sqrt{a_1 \cdots a_k A_1 \cdots A_k}\right]^2}. \tag{3.2}$$

Similarly, we define

$$g_i(b_{1j}) = b_{ij}, \qquad \eta_i = g_i(\eta_1), \quad i = 2, \dots, t; \ j = 1, \dots, m,$$

then

$$E\eta_i^2 = \frac{1}{m}\sum_{j=1}^{m} b_{ij}^2, \quad i = 1, \dots, t,$$

$$E(\eta_1 \cdots \eta_t) = \frac{1}{m}\sum_{j=1}^{m}\prod_{i=1}^{t} b_{ij},$$

$$\overline{A}(\eta_1, \dots, \eta_k) = \frac{1}{2}(b_1 \cdots b_k + B_1 \cdots B_k),$$

$$\overline{G}(\eta_1, \dots, \eta_k) = \sqrt{b_1 \cdots b_k B_1 \cdots B_k},$$

$$E(\xi_1 \cdots \xi_n \eta_1 \cdots \eta_t) = \frac{1}{m}\sum_{j=1}^{m}\prod_{i=1}^{n} a_{ij} b_{ij}.$$

Using inequality (1.1) again gets

$$\frac{E\eta_1^2 \cdots E\eta_t^2}{E^2(\eta_1 \cdots \eta_t)} \le \prod_{k=2}^{t} \frac{\left[\frac{1}{2}(b_1 \cdots b_k + B_1 \cdots B_k)\right]^2}{\left[\sqrt{b_1 \cdots b_k B_1 \cdots B_k}\right]^2}. \tag{3.3}$$

From inequality (2.5), we have

$$\begin{aligned} \frac{E\xi_1^2 \cdots E\xi_n^2 \cdot E\eta_1^2 \cdots E\eta_t^2}{E^2(\xi_1 \cdots \xi_n \cdot \eta_1 \cdots \eta_t)} \le\ & \frac{\overline{A}^2(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m)}{\overline{G}^2(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m)} \\ & \times \prod_{k=2}^{n} \frac{\left[\frac{1}{2}(a_1 \cdots a_k + A_1 \cdots A_k)\right]^2}{\left[\sqrt{a_1 \cdots a_k A_1 \cdots A_k}\right]^2} \prod_{k=2}^{t} \frac{\left[\frac{1}{2}(b_1 \cdots b_k + B_1 \cdots B_k)\right]^2}{\left[\sqrt{b_1 \cdots b_k B_1 \cdots B_k}\right]^2}. \end{aligned} \tag{3.4}$$

That is

$$\frac{\prod_{i=1}^{n}\left(\frac{1}{m}\sum_{j=1}^{m}a_{ij}^2\right)\cdot\prod_{i=1}^{t}\left(\frac{1}{m}\sum_{j=1}^{m}b_{ij}^2\right)}{\left(\frac{1}{m}\sum_{j=1}^{m}\prod_{i=1}^{n}a_{ij}b_{ij}\right)^2} \leq \frac{\left[\frac{1}{2}\left(\prod_{i=1}^{n}a_i\prod_{i=1}^{t}b_i+\prod_{i=1}^{n}A_i\prod_{i=1}^{t}B_i\right)\right]^2}{\left[\sqrt{\prod_{i=1}^{n}a_iA_i\prod_{i=1}^{t}b_iB_i}\right]^2}$$

$$\times\prod_{k=2}^{n}\frac{\left[\frac{1}{2}(a_1\cdots a_k+A_1\cdots A_k)\right]^2}{\left[\sqrt{a_1\cdots a_kA_1\cdots A_k}\right]^2}\prod_{k=2}^{t}\frac{\left[\frac{1}{2}(b_1\cdots b_k+B_1\cdots B_k)\right]^2}{\left[\sqrt{b_1\cdots b_kB_1\cdots B_k}\right]^2}, \tag{3.5}$$

from which the result easily follows. □

Remark 3.2. If $n = t = 2$, this inequality can be expressed as

$$\left(\sum_{i=1}^{m}a_i^2\right)\left(\sum_{i=1}^{m}b_i^2\right)\left(\sum_{i=1}^{m}c_i^2\right)\left(\sum_{i=1}^{m}d_i^2\right) \leq \frac{m^2}{64}\frac{(abcd+ABCD)^2}{abcdABCD}\frac{(ab+AB)^2}{abAB}\frac{(cd+CD)^2}{cdCD}\left(\sum_{i=1}^{m}a_ib_ic_id_i\right)^2, \tag{3.6}$$

where $a_i, b_i, c_i, d_i > 0$ for $i = 1, \ldots, m$, and

$$a=\min a_i,\qquad A=\max a_i,\qquad b=\min b_i,\qquad B=\max b_i,\qquad c=\min c_j,$$
$$C=\max c_j,\qquad d=\min d_j,\qquad D=\max d_j.$$

Then we have the inequity of multiple integral.

Theorem 3.3. *Let $f_i(x),\ i = 1, \ldots, n$ and $g_j(x),\ j = 1, \ldots, m$ be continuous functions on $[a, b]$. Let $\phi(x, y)$ be a non-negative integral function on $[a, b]^2$ satisfying*

$$\int_a^b\int_a^b\phi(x,y)\mathrm{d}x\mathrm{d}y = 1.$$

Let

$$a_i=\inf_{x\in[a,b]}f_i(x),\qquad A_i=\sup_{x\in[a,b]}f_i(x),\quad i=1,\ldots,n$$
$$b_j=\inf_{x\in[a,b]}g_j(x),\qquad B_j=\sup_{x\in[a,b]}g_j(x),\quad j=1,\ldots,m.$$

If $a_i, b_j > 0$ and $m, n \geq 2$ then

$$\prod_{i=1}^{n}\int_a^b\int_a^b f_i^2(x)\phi(x,y)\mathrm{d}x\mathrm{d}y\cdot\prod_{j=1}^{m}\int_a^b\int_a^b g_j^2(x)\phi(x,y)\mathrm{d}x\mathrm{d}y \leq \frac{1}{4^{n+m-1}}\cdot\frac{[a_1\cdots a_nb_1\cdots b_m+A_1\cdots A_nB_1\cdots B_m]^2}{a_1\cdots a_nb_1\cdots b_mA_1\cdots A_nB_1\cdots B_m}$$

$$\times\prod_{k=2}^{n}\frac{(a_1\cdots a_k+A_1\cdots A_k)^2}{a_1\cdots a_kA_1\cdots A_k}\cdot\prod_{k=2}^{m}\frac{(b_1\cdots b_k+B_1\cdots B_k)^2}{b_1\cdots b_kB_1\cdots B_k}\left[\int_a^b\int_a^b\prod_{i=1}^{n}f_i(x)\cdot\prod_{j=1}^{m}g_j(y)\cdot\phi(x,y)\mathrm{d}x\mathrm{d}y\right]^2. \tag{3.7}$$

Proof. We define the random variable (ξ, η) by the joint probability density function $\phi(x, y)$. Let $\xi_i = f_i(\xi),\ i = 1, \ldots, n$. Then

$$E\xi_i^2=\int_a^b\int_a^b f_i^2(x)\phi(x,y)\mathrm{d}x\mathrm{d}y,$$
$$E\left(\prod_{i=1}^{n}\xi_i\right)=\int_a^b\int_a^b\prod_{i=1}^{n}f_i(x)\phi(x,y)\mathrm{d}x\mathrm{d}y,$$
$$\overline{A}(\xi_1,\ldots,\xi_k)=\frac{1}{2}(a_1\cdots a_k+A_1\cdots A_k),$$
$$\overline{G}(\xi_1,\ldots,\xi_k)=\sqrt{a_1\cdots a_kA_1\cdots A_k},\quad k=2,\ldots,n.$$

Using inequality (1.1), we have

$$\frac{E\xi_1^2\cdots E\xi_n^2}{E^2(\xi_1\cdots\xi_n)} \leq \prod_{k=2}^{n}\frac{\left[\frac{1}{2}(a_1\cdots a_k+A_1\cdots A_k)\right]^2}{\left[\sqrt{a_1\cdots a_kA_1\cdots A_k}\right]^2}. \tag{3.8}$$

Now let $\eta_j = g_j(\eta),\ j = 1, \ldots, m$, then

$$E\eta_j^2 = \int_a^b \int_a^b g_j^2(x)\phi(x,y)\mathrm{d}x\mathrm{d}y,$$
$$E\left(\prod_{j=1}^m \eta_j\right) = \int_a^b \int_a^b \prod_{j=1}^m g_j(x)\phi(x,y)\mathrm{d}x\mathrm{d}y,$$
$$\overline{A}(\eta_1, \ldots, \eta_k) = \frac{1}{2}(b_1 \cdots b_k + B_1 \cdots B_k),$$
$$\overline{G}(\eta_1, \ldots, \eta_k) = \sqrt{b_1 \cdots b_k B_1 \cdots B_k}, \quad k = 2, \ldots, m,$$
$$E(\xi_1 \cdots \xi_n \eta_1 \cdots \eta_m) = \int_a^b \int_a^b \prod_{i=1}^n f_i(x) \cdot \prod_{j=1}^m g_j(y)\phi(x,y)\mathrm{d}x\mathrm{d}y.$$

Using inequality (1.1) again obtains

$$\frac{E\eta_1^2 \cdots E\eta_m^2}{E^2(\eta_1 \cdots \xi_m)} \le \prod_{k=2}^m \frac{\left[\frac{1}{2}(b_1 \cdots b_k + B_1 \cdots B_k)\right]^2}{\left[\sqrt{b_1 \cdots b_k B_1 \cdots B_k}\right]^2}. \tag{3.9}$$

Now we employ inequality (2.5) to conclude that

$$\begin{aligned}\frac{E\xi_1^2 \cdots E\xi_n^2 \cdot E\eta_1^2 \cdots E\eta_m^2}{E^2(\xi_1 \cdots \xi_n \cdot \eta_1 \cdots \eta_m)} &\le \frac{\overline{A}^2(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m)}{\overline{G}^2(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m)} \prod_{k=2}^n \frac{\left[\frac{1}{2}(a_1 \cdots a_k + A_1 \cdots A_k)\right]^2}{\left[\sqrt{a_1 \cdots a_k A_1 \cdots A_k}\right]^2} \\ &\quad \times \prod_{k=2}^m \frac{\left[\frac{1}{2}(b_1 \cdots b_k + B_1 \cdots B_k)\right]^2}{\left[\sqrt{b_1 \cdots b_k B_1 \cdots B_k}\right]^2}\end{aligned} \tag{3.10}$$

which implies the inequality we seek. $\square$

Remark 3.4. If $n = m = 2$, this inequality has the following expression:

$$\begin{aligned}&\int_a^b \int_a^b f_1^2(x)\phi(x,y)\mathrm{d}x\mathrm{d}y \int_a^b \int_a^b f_2^2(x)\phi(x,y)\mathrm{d}x\mathrm{d}y \int_a^b \int_a^b f_3^2(x)\phi(x,y)\mathrm{d}x\mathrm{d}y \int_a^b \int_a^b f_4^2(x)\phi(x,y)\mathrm{d}x\mathrm{d}y \\ &\quad \le \frac{1}{64} \frac{(a_1a_2a_3a_4 + A_1A_2A_3A_4)^2}{a_1a_2a_3a_4A_1A_2A_3A_4} \frac{(a_1a_2 + A_1A_2)^2}{a_1a_2A_1A_2} \frac{(a_3a_4 + A_3A_4)^2}{a_3a_4A_3A_4} \left[\int_a^b \int_a^b f_1(x)f_2(x)f_3(y)f_4(y)\phi(x,y)\mathrm{d}x\mathrm{d}y\right]^2,\end{aligned}$$

where $a_i = \inf_{x\in[a,b]} f_i(x)$, $A_i = \sup_{x\in[a,b]} f_i(x)$, $i = 1, \ldots, 4$.

Finally, we give the following generalizations of the *Greub–Rheinboldt* inequality [9].

Theorem 3.5 (*The Extensions of the Greub–Rheinboldt Inequality*)**.** *Let A and B be two $m \times m$ positive Hermitian matrices, and suppose $AB = BA$. Let $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_m$ denote the eigenvalues of A and B, respectively. For $\beta = \beta_1 + \cdots + \beta_s$ and $\gamma = \gamma_1 + \cdots + \gamma_t$ where β_i and γ_j are real, and any vector $x \neq 0$, the following inequality holds:*

$$\begin{aligned}\frac{\prod_{i=1}^s x^*A^{\beta_i}x \prod_{i=1}^t x^*B^{\gamma_i}x}{(x^*A^{\beta/2}B^{\gamma/2}x)^2} &\le \frac{(x^*x)^{s+t-2}}{4^{s+t-1}} \frac{[a_1 \cdots a_s b_1 \cdots b_t + A_1 \cdots A_s B_1 \cdots B_t]^2}{a_1 \cdots a_s b_1 \cdots b_t A_1 \cdots A_s B_1 \cdots B_t} \\ &\quad \times \prod_{k=2}^s \frac{[a_1 \cdots a_k + A_1 \cdots A_k]^2}{a_1 \cdots a_k A_1 \cdots A_k} \prod_{k=2}^t \frac{[b_1 \cdots b_k + B_1 \cdots B_k]^2}{b_1 \cdots b_k B_1 \cdots B_k},\end{aligned} \tag{3.11}$$

where

$$a_i = \begin{cases} \lambda_{(m)}^{\beta_i/2}, & \beta_i \ge 0 \\ \lambda_{(1)}^{\beta_i/2}, & \beta_i < 0; \end{cases} \qquad A_i = \begin{cases} \lambda_{(1)}^{\beta_i/2}, & \beta_i \ge 0 \\ \lambda_{(m)}^{\beta_i/2}, & \beta_i < 0, \end{cases} \qquad i = 1, \ldots, s$$
$$b_j = \begin{cases} \mu_{(m)}^{\gamma_j/2}, & \gamma_j \ge 0 \\ \mu_{(1)}^{\gamma_j/2}, & \gamma_j < 0; \end{cases} \qquad B_j = \begin{cases} \mu_{(1)}^{\gamma_j/2}, & \gamma_j \ge 0 \\ \mu_{(m)}^{\gamma_j/2}, & \gamma_j < 0, \end{cases} \qquad j = 1, \ldots, t.$$

$\lambda_{(1)} = \max \lambda_i$, $\lambda_{(m)} = \min \lambda_i$, $\mu_{(1)} = \max \mu_i$, $\mu_{(m)} = \min \mu_i$.

Proof. Since $AB = BA$, there exists a Hermitian matrix U that satisfies $A = U^*\Lambda U$ and $B = U^*MU$, where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$ and $M = \mathrm{diag}(\mu_1, \dots, \mu_m)$.

Let

$$Y = Ux = (y_1, y_2, \dots, y_m)^T,$$
$$p_i = \frac{|y_i|^2}{\sum_{i=1}^{n} |y_i|^2}, \quad i = 1, \dots, m,$$

then

$$\begin{aligned}\frac{\prod_{i=1}^{s} x^*A^{\beta_i}x \prod_{i=1}^{t} x^*B^{\gamma_i}x}{(x^*A^{\beta/2}B^{\gamma/2}x)^2} &= \frac{\prod_{i=1}^{s} x^*U^*\Lambda^{\beta_i}Ux \prod_{i=1}^{t} x^*U^*M^{\gamma_i}Ux}{(x^*U^*\Lambda^{\frac{\beta}{2}}M^{\frac{\gamma}{2}}Ux)^2}\\
&= \frac{\prod_{i=1}^{s} y^*\Lambda^{\beta_i}y \prod_{i=1}^{t} y^*M^{\gamma_i}y}{(y^*\Lambda^{\frac{\beta}{2}}M^{\frac{\gamma}{2}}y)^2}\\
&= \frac{(y^*y)^{s+t-2}\prod_{i=1}^{s}\left(\sum_{j=1}^{m}\lambda_j^{\beta_i}p_j\right)\prod_{i=1}^{t}\left(\sum_{j=1}^{m}\mu_j^{\gamma_i}p_j\right)}{\left(\sum_{j=1}^{m}\lambda_j^{\frac{\beta}{2}}\mu_j^{\frac{\gamma}{2}}p_j\right)^2}\\
&= \frac{(x^*x)^{s+t-2}\prod_{i=1}^{s}\left(\sum_{j=1}^{m}\lambda_j^{\beta_i}p_j\right)\prod_{i=1}^{t}\left(\sum_{j=1}^{m}\mu_j^{\gamma_i}p_j\right)}{\left(\sum_{j=1}^{m}\lambda_j^{\frac{\beta}{2}}\mu_j^{\frac{\gamma}{2}}p_j\right)^2}.\end{aligned} \tag{3.12}$$

Our problem reduce to proving

$$\begin{aligned}\frac{\prod_{i=1}^{s}\left(\sum_{j=1}^{m}\lambda_j^{\beta_i}p_j\right)\prod_{i=1}^{t}\left(\sum_{j=1}^{m}\mu_j^{\gamma_i}p_j\right)}{\left(\sum_{j=1}^{m}\lambda_j^{\frac{\beta}{2}}\mu_j^{\frac{\gamma}{2}}p_j\right)^2} &\le \frac{1}{4^{s+t-1}}\frac{[a_1\cdots a_sb_1\cdots b_t + A_1\cdots A_sB_1\cdots B_t]^2}{a_1\cdots a_sb_1\cdots b_tA_1\cdots A_sB_1\cdots B_t}\\
&\quad\times\prod_{k=2}^{s}\frac{[a_1\cdots a_k + A_1\cdots A_k]^2}{a_1\cdots a_kA_1\cdots A_k}\prod_{k=2}^{t}\frac{[b_1\cdots b_k + B_1\cdots B_k]^2}{b_1\cdots b_kB_1\cdots B_k}.\end{aligned} \tag{3.13}$$

We define the two random variables by the following joint distribution of (ξ, η):

$$p\left\{(\xi, \eta) = (\lambda_i, \mu_j)\right\} = \begin{cases} p_i, & \text{when } i = j,\\ 0, & \text{otherwise.}\end{cases}$$

Then the probability distribution of ξ is

$$p(\xi = \lambda_i) = p_i, \quad i = 1, \dots, m,$$

and the probability distribution of η is

$$p(\eta = \mu_i) = p_i, \quad i = 1, \dots, m.$$

Suppose $\xi_i = \xi^{\frac{\beta_i}{2}}, i = 1, \dots, s$; and $\eta_i = \eta^{\frac{\gamma_i}{2}},\ i = 1, \dots, t$, we have

$$\prod_{i=1}^{s}\left(\sum_{j=1}^{m}\lambda_j^{\beta_i}p_j\right) = \prod_{i=1}^{s}E\xi_i^2,$$
$$\prod_{i=1}^{t}\left(\sum_{j=1}^{m}\mu_j^{\gamma_i}p_j\right) = \prod_{i=1}^{t}E\eta_i^2, \qquad \sum_{j=1}^{m}\lambda a_j^{\frac{\beta}{2}}\mu_j^{\frac{\gamma}{2}}p_j = E\left(\prod_{i=1}^{s}\xi_i\prod_{i=1}^{t}\eta_i\right),$$

$$\frac{\prod_{i=1}^{s}\left(\sum_{j=1}^{m}\lambda_j^{\beta_i}p_j\right)\prod_{i=1}^{t}\left(\sum_{j=1}^{m}\mu_j^{\gamma_i}p_j\right)}{\left(\sum_{j=1}^{m}\lambda_j^{\frac{\beta}{2}}\mu_j^{\frac{\gamma}{2}}p_j\right)^2}=\frac{\prod_{i=1}^{s}E\xi_i^2\prod_{i=1}^{t}E\eta_i^2}{E^2\left(\prod_{i=1}^{s}\xi_i\prod_{i=1}^{t}\eta_i\right)}.$$

According to inequality (2.5)

$$\frac{\prod_{i=1}^{n}E\xi_i^2\prod_{j=1}^{m}E\eta_j^2}{E^2\left(\prod_{i=1}^{n}\xi_i\prod_{j=1}^{m}\eta_m\right)}\le\frac{\overline{A}^2(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m)}{\overline{G}^2(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m)}\prod_{k=1}^{n}\frac{\overline{A}^2(\xi_1,\ldots,\xi_k)}{\overline{G}^2(\xi_1,\ldots,\xi_k)}\prod_{k=1}^{m}\frac{\overline{A}^2(\eta_1,\ldots,\eta_k)}{\overline{G}^2(\eta_1,\ldots,\eta_k)}. \tag{3.14}$$

Using Lemma 2.1 gives

$$\begin{aligned}\frac{\prod_{i=1}^{n}E\xi_i^2\prod_{j=1}^{m}E\eta_j^2}{E^2\left(\prod_{i=1}^{n}\xi_i\prod_{j=1}^{m}\eta_m\right)}&\le\frac{\left[\frac{1}{2}(a_1\cdots a_sb_1\cdots b_t+A_1\cdots A_sB_1\cdots B_t)\right]^2}{a_1\cdots a_sb_1\cdots b_t+A_1\cdots A_sB_1\cdots B_t}\\&\quad\times\prod_{k=2}^{s}\frac{\left[\frac{1}{2}(a_1\cdots a_k+A_1\cdots A_k)\right]^2}{\left[\sqrt{a_1\cdots a_kA_1\cdots A_k}\right]^2}\prod_{k=2}^{t}\frac{\left[\frac{1}{2}(b_1\cdots b_k+B_1\cdots B_k)\right]^2}{\left[\sqrt{b_1\cdots b_kB_1\cdots B_k}\right]^2}\\&=\frac{1}{4^{s+t-1}}\frac{[a_1\cdots a_sb_1\cdots b_t+A_1\cdots A_sB_1\cdots B_t]^2}{a_1\cdots a_sb_1\cdots b_t+A_1\cdots A_sB_1\cdots B_t}\prod_{k=2}^{s}\frac{[a_1\cdots a_k+A_1\cdots A_k]^2}{\left[\sqrt{a_1\cdots a_kA_1\cdots A_k}\right]^2}\\&\quad\times\prod_{k=2}^{t}\frac{(b_1\cdots b_k+B_1\cdots B_k)^2}{\left[\sqrt{b_1\cdots b_kB_1\cdots B_k}\right]^2},\end{aligned} \tag{3.15}$$

from which we conclude (3.11). □

Remark 3.6. If $s=t=1$ the inequality can be expressed by

$$\frac{x^*A^2xx^*B^2x}{(x^*ABx)^2}\le\frac{(\lambda_1\mu_1+\lambda_n\mu_n)^2}{4\lambda_1\lambda_n\mu_1\mu_n}, \tag{3.16}$$

this inequality is the *Greub–Rheinboldt* inequality [9].

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